PHYS 304 Supplemental Notes.

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Goal is to see classical-like oscillitory behaviour in $|\Psi(x,t)|^2$ of a quantum simple harmonic motion.

Know

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad E_n = \hbar\omega\left(n+\frac{1}{2}\right)$$

$$\Psi(x,0) = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\psi_n(x)$$

will take $\alpha = 0+i6$

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}\, H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

Hermite polynomials

$$H_n(\xi) = (-1)^n e^{\xi^2}\frac{d^{(n)}}{d\xi^n}\left(e^{-\xi^2}\right)$$

try $H_0 = e^{\xi^2} e^{-\xi^2} = 1$

$$H_1 = -e^{\xi^2} \frac{d}{d\xi} e^{-\xi^2} = -e^{\xi^2} (-2\xi) e^{-\xi^2}$$
$$= 2\xi$$

$$H_2 = e^{\xi^2} \frac{d^2}{d\xi^2} e^{-\xi^2} = e^{\xi^2} \frac{d}{d\xi} \left(-2\xi e^{-\xi^2} \right)$$
$$= e^{\xi^2} \left[-2e^{-\xi^2} + 4\xi^2 e^{-\xi^2} \right]$$
$$= 2\left[-1 + 2\xi^2 \right]$$

$$H_3 = -e^{\xi^2} \frac{d}{d\xi} \left[-2e^{-\xi^2} + 4\xi^2 e^{-\xi^2} \right]$$
$$= -e^{\xi^2} \left[4\xi e^{-\xi^2} + 8\xi e^{-\xi^2} - 8\xi^3 e^{-\xi^2} \right]$$
$$= -\left[12\xi - 8\xi^3 \right] = 4\xi \left[2\xi^2 - 3 \right]$$

$\vdots$

$$\Psi(x,t) = \sum_{n=0}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right)$$

$$\therefore E_n t/\hbar = \omega t\left(n+\frac{1}{2}\right)$$

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2} H_n(\xi)$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$$

take $c_n = e^{-|\alpha|^2/2}\dfrac{\alpha^n}{\sqrt{n!}}$

$$\therefore \Psi(x,t) = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\xi^2/2} H_n(\xi)$$

$$\cdot\, e^{-i\omega t\left(n+\frac{1}{2}\right)}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-|\alpha|^2/2} e^{-i\omega t/2} e^{-\xi^2/2}\sum_{n=0}^{\infty}\frac{H_n(\xi)}{n!}\left(\frac{\alpha e^{-i\omega t}}{\sqrt{2}}\right)^n$$

use $$\sum_{n=0}^{\infty} \frac{H_n(\xi)}{n!} s^n = \exp(2\xi s - s^2) \quad \#$$

Proof of #

Start w/ generating fcn

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^{(n)}}{d\xi^{(n)}} e^{-\xi^2}$$

$$\therefore \sum_{n=0}^{\infty} \frac{H_n(\xi)}{n!} s^n = e^{\xi^2} \sum_{n=0}^{\infty} \frac{(-1)^n s^n}{n!} \frac{d^{(n)}}{d\xi^{(n)}} e^{-\xi^2}$$

but $$\sum_{n=0}^{\infty} \frac{\left[(-s)^n \frac{d^{(n)}}{d\xi^{(n)}}\right]}{n!} = e^{-s\frac{d}{d\xi}}$$

$$= e^{\xi^2} \exp\left(-s\frac{d}{d\xi}\right) e^{-\xi^2}$$

Use the identity $\exp\left(-s\frac{d}{d\xi}\right) f(\xi) = f(\xi - s)$ #

Proof of (#)

Consider $f(z)$ and expand about ξ

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-\xi)^n}{n!} \underbrace{\left.\frac{d^n f(z)}{dz^n}\right|_{z=\xi}}_{\text{equiv. to } \frac{d^n f(\xi)}{d\xi^n}}$$

Now let $z = \xi - s$

$$\therefore f(\xi - s) = \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} \frac{d^n f(\xi)}{d\xi^n} = \exp\left(-s\frac{d}{d\xi}\right)$$

$$\therefore \exp\left(-s\frac{d}{d\xi}\right) e^{-\xi^2} = e^{-(\xi-s)^2}$$

$$\therefore \sum_{n=0}^{\infty} \frac{H_n(\xi)}{n!} s^n = \cancel{e^{\xi^2}}\, e^{-(\cancel{\xi^2} - 2\xi s + s^2)}$$

$$= e^{2\xi s - s^2} \checkmark$$

$$\Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-|\alpha|^2/2} e^{-i\omega t/2} e^{-\xi^2/2} \sum_{n=0}^{\infty} \frac{H_n(\xi)}{n!} \Big(\underbrace{\frac{\alpha e^{-i\omega t}}{\sqrt{2}}}_{\equiv s}\Big)^n$$

becomes:

$$\Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-|\alpha|^2/2} e^{-i\omega t/2} e^{-\xi^2/2} e^{2\xi s - s^2}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-\frac{|\alpha|^2}{2} - \frac{i\omega t}{2} - \frac{\xi^2}{2} + 2\xi s - s^2\right]$$

$$\Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-\frac{|\alpha|^2}{2} - \frac{i\omega t}{2} - \frac{\xi^2}{2} + 2\xi s - s^2\right]$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x \qquad s = \frac{\alpha e^{-i\omega t}}{\sqrt{2}}$$

$$-\frac{|\alpha|^2}{2} - \frac{i\omega t}{2} - \frac{1}{2}\left(\xi^2 - 4\xi s + 2s^2 + 2s^2\right) + s^2$$

$$= -\frac{|\alpha|^2}{2} - \frac{i\omega t}{2} + s^2 - \frac{1}{2}\underbrace{\left(\xi - 4\xi s + (2s)^2\right)}_{(\xi - 2s)^2}$$

$$= -\frac{|\alpha|^2}{2} - \frac{i\omega t}{2} + s^2 - \frac{1}{2}(\xi - 2s)^2$$

write $s = u + iv$ where $u = \frac{\mathrm{Re}\left[\alpha e^{-i\omega t}\right]}{\sqrt{2}}$

$v = \frac{\mathrm{Im}\left[\alpha e^{-i\omega t}\right]}{\sqrt{2}}$

$$= -\frac{|\alpha|^2}{2} - \frac{i\omega t}{2} + u^2 + 2iuv - v^2 - \frac{1}{2}\left((\xi - 2u) - 2iv\right)^2$$

$$= -\frac{|\alpha|^2}{2} - \frac{i\omega t}{2} + u^2 + 2iuv - v^2 - \frac{1}{2}\left[(\xi - 2u)^2 - 4iv(\xi - 2u) - 4v^2\right]$$

$$= -\frac{|\alpha|^2}{2} - \frac{i\omega t}{2} + u^2 - v^2 + 2iuv - \frac{1}{2}(\xi - 2u)^2 + 2iv(\xi - 2u) + 2v^2$$

$$\alpha = \sqrt{2}\, s e^{i\omega t}$$

$$\therefore \frac{|\alpha|^2}{2} = \frac{2|s|^2}{2} = |s|^2 = u^2 + v^2$$

$$= \cancel{-u^2} \cancel{-v^2} - \frac{i\omega t}{2} \cancel{+u^2} \cancel{-v^2} + 2iuv + \cancel{2v^2} - \frac{1}{2}(\xi - 2u)^2 + 2iv(\xi - 2u)$$

$$= -\frac{1}{2}(\xi - 2u)^2 + \underline{2iv}(\xi - \underline{2u}) + \underline{i}(\underline{2uv} - \omega t/2)$$

combine to make

$$-4iuv + 2iuv = -2iuv$$

$$= -\frac{1}{2}(\xi - 2u)^2 + 2iv\xi - 2iuv - i\omega t/2$$

$$= -\frac{1}{2}(\xi - 2u)^2 + 2iv(\xi - u) - i\omega t/2$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\hookrightarrow = -\frac{1}{2}\frac{m\omega}{\hbar}\left(x - 2\sqrt{\frac{\hbar}{m\omega}}\,u\right)^2 + 2i\sqrt{\frac{m\omega}{\hbar}}\,v\left(x - \sqrt{\frac{\hbar}{m\omega}}\,u\right) - \frac{i\omega t}{2}$$

$$2\sqrt{\frac{\hbar}{m\omega}}\,u = 2\sqrt{\frac{\hbar}{m\omega}}\,\frac{\mathrm{Re}\left[\alpha e^{-i\omega t}\right]}{\sqrt{2}}$$

$$= \boxed{\sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\left[\alpha e^{-i\omega t}\right] \equiv x_c}$$

$$\hookrightarrow = -\frac{m\omega}{2\hbar}\left(x - x_c\right)^2 + 2i\sqrt{\frac{m\omega}{\hbar}}\,v\left(x - \frac{x_c}{2}\right) - \frac{i\omega t}{2}$$

$$2\sqrt{\frac{m\omega}{\hbar}}\,v = \frac{\cancel{\sqrt{2}}}{\hbar}\sqrt{2m\hbar\omega}\,\frac{\mathrm{Im}\left[\alpha e^{-i\omega t}\right]}{\cancel{\sqrt{2}}}$$

$$= \boxed{\frac{1}{\hbar}\sqrt{2m\hbar\omega}\,\mathrm{Im}\left[\alpha e^{-i\omega t}\right] \equiv \frac{p_c(t)}{\hbar}}$$

$$\hookrightarrow = \boxed{-\frac{m\omega}{2\hbar}(x-x_c)^2 + \frac{i}{\hbar}p_c\left(x-\frac{x_c}{2}\right) - \frac{i\omega t}{2}}$$

$$\therefore \Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}(x-x_c(t))^2\right]$$

$$\cdot \exp\left[\frac{i}{\hbar}p_c(t)\left(x-\frac{x_c(t)}{2}\right)\right]$$

$$\cdot e^{-i\omega t/2}$$

where $x_c(t) = \sqrt{\frac{2\hbar}{m\omega}}\, \mathrm{Re}\left[\alpha e^{-i\omega t}\right]$

$$p_c(t) = \sqrt{2m\hbar\omega}\, \mathrm{Im}\left[\alpha e^{-i\omega t}\right]$$

If I take $\alpha = 6i$

$$\mathrm{Re}\left[\alpha e^{-i\omega t}\right] = \mathrm{Re}\left[6i(\cos\omega t - i\sin\omega t)\right]$$

$$= \mathrm{Re}\left[6i\cos\omega t + 6\sin\omega t\right]$$

$$= 6\sin\omega t \rightarrow x_c = 6\sqrt{\frac{2\hbar}{m\omega}}\sin\omega t$$

$$\mathrm{Im}\left[\alpha e^{-i\omega t}\right] = 6\cos\omega t \rightarrow p_c = 6\sqrt{2m\hbar\omega}\cos\omega t$$

$$\therefore |\Psi(x,t)|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left[-\frac{m\omega}{\hbar}\left(x - x_c(t)\right)^2\right]$$

For $\alpha = 6i$

$$|\Psi(x,t)|^2 = \overbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}}^{\frac{1}{\sqrt{2\pi}\,\sigma}}\exp\left[\overbrace{-\frac{m\omega}{\hbar}}^{\frac{1}{2\sigma^2}}\left(x - \overbrace{6\underbrace{\sqrt{\frac{2\hbar}{m\omega}}}_{2\sigma}\sin\omega t}^{\mu(t)}\right)^2\right]$$

The quantum probability density is a Gaussian of mean $\mu(t)$ that oscillates about zero w/ amplitude prop. to $|\alpha|$ and freq. ω.

$$P_G \propto e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$\therefore$ The width σ of our oscillating Gaussian pulse is:

$$\frac{1}{2\sigma^2} = \frac{m\omega}{\hbar}$$

$$\therefore \boxed{\sigma = \sqrt{\frac{\hbar}{2m\omega}}}$$

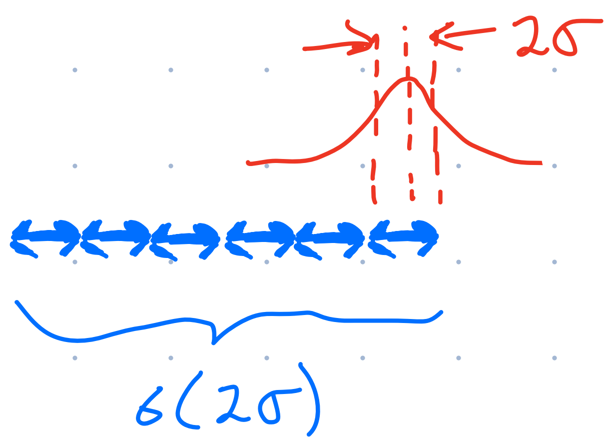
Check units:

$$\left[\frac{\hbar}{m\omega}\right] = \frac{J\,s}{kg\,\frac{1}{s}} = \frac{N m s^2}{kg} = \frac{\frac{\cancel{kg}\,m\,m\,\cancel{s^2}}{\cancel{s^2}}}{\cancel{kg}} = m^2$$

$$\therefore [\sigma] = m \quad \checkmark$$

Notice that the amplitude of the oscillation is also prop. to 2σ, while the amplitude of the Gaussian pulse itself, is equal to $\frac{1}{\sqrt{2\pi}\,\sigma}$.

So, when I chose $\alpha = 6i$, the factor of 6 ultimately sets how many multiples of 2σ the mean of our Gaussian pulse will oscillate through.



This is the amplitude of the oscillation.
i.e. 1/2 of the "peak-to-peak" motion.